Proof of the Jordan curve theorem

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Abstract

In the following we will represent the Jordan curve theorem in the form and generality needed during the course Function theory III lectured in the fall of 2010 at University of Helsinki. We will prove the Jordan curve theorem in two ways, one being an elementary proof and the other using the Brouwer fixed point theorem, which is also proven.

All proofs are done in the spirit of elementary complex analysis and this essay is meant to be largely self-contained, although some prequisites of higher complex analysis are required. References are given to the somewhat large variety of different proofs of the theorem.

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1 Introduction

We will begin by going through some notions on the history of the theorem and its proofs and a summary of notations, basic consepts and the goal of this essay.

1.1 The theorem

The Jordan curve theorem states the following:

Theorem 1.1 (The Jordan curve theorem, abbreviated JCT)**.** *The image of a continuous injective mapping (i.e. an embedding)*

$$J\colon S^1 \to \mathbb{R}^2$$

divides the plane into exactly two components, one of which is unbounded and the other bounded. Moreover, both of these components have the image of the mapping J as their boundary.

The theorem was first formulated, at least in some form, by *Bernand Bolzano (1781-1848)* but it is named after the french mathematician *Camille Jordan (1838-1922)*, for he was the first to publish a proof for the theorem in 1887 at [Jo]. The validity of his proof was questioned by his contemporaries, but some controversy has risen concerning whether or not the criticism was justified, see [Hal]. First generally accepted rigorous proof was given by the american mathematician *Oswald Veblen (1880-1960)* at [Ve] in 1905. Also the Dutch mathematician *Jan Brouwer (1981-1966)*, famous for e.g. the Brouwer fixed point theorem, worked with the Jordan curve theorem, and managed to prove one of its generalizations with *Henri Lebesgue (1875-1941)*. (We will return to this later.)

The JCT is at the same time famous and notorious for being both *really* intuitive and *quite* nontrivial to prove rigorously. In someone elses words:

> "This is the mathematical formulation of a fact that shepherds have relied on since time immemorial!"
> – Laurent Siebenmann

In fact Camille Jordan was the first, according to [DT], to notice and discuss the nontriviality of the theorem in written form in [Jo].

Not surprisingly, the combination of being highly intuitive and lacking a trivial proof has given rise to a horde of mathematicians creating a large amount of proofs with quite different types of approaches. List of some of these with references will be given at section 3.

To paint a more complete picture of the history and the structure of the JCT, I would like to finish the historical section by saying a few words about the generalizations of the JCT. Before going to generalizations of the JCT, however, we note that there exists also a "strong" version of the JCT, called the Schöenflies- or Jordan-Schöenflies theorem. This is not as such a generalization of the JCT as a strenghtened argument. It states the following.

Theorem (The Jordan-Schöenflies theorem)**.** *Given an embedding c of the unit sphere S^1 to the plane $\mathbb{R}^2$, there exists a homeomorphism*

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2$$

such that $f|_{S^1} = c$.

The JCT can be generalized into all dimensions $n \geq 2$ in the form of the following theorem:

Theorem (The Jordan-Brouwer theorem)**.** *The image of an embedding*

$$f\colon S^n \to \mathbb{R}^{n+1}$$

divides the $n+1$*-dimensional euclidian space into exactly two components, one of which is unbounded and the other bounded. Moreover, both of these components have the image of the mapping* f *as their boundary.*

As previously mentioned, the Jordan-Brouwer theorem was first proved in 1911 by J. Brouwer and H. Lebesgue.[2]

One reason I wanted to mention the Jordan-Schöenflies theorem is, that unlike the JCT it quite surprisingly cannot be generalized even into the case of dimension 3. A famous counter-example of an embedding of S^2 into $\mathbb{R}^3$ with the property that the unbounded components of the complements of S^2 and its image are not homemorphic is the so-called *Alexander horned sphere*, first defined by *J.W. Alexander (1888-1971)* in his paper [Al][3] in 1924.

1.2 Notations and basic concepts

General

We shall abbreviate the Jordan Curve Theorem in the form given in theorem 1.1 by JCT.

"Our course" or "Function theory III" and "Lecture notes" will refer, unless otherwise specified, to the course *Function Theory III* lectured by Eero Saksman at the University of Helsinki in the fall of 2010, and to the lecture notes [FT3] used in that course.

By "Function Theory II" we refer to the course *Function Theory II* also lectured by Eero Saksman at the University of Helsinki in the spring of 2010. The contents of this course can be found from the lecture notes [FT2] used in that course.

Topology

By a *domain* we will mean an open connected subset of the complex plane. We often note a domain by Ω.

We say that a *set* A *separates points* $a, b \in \mathbb{C} \setminus A$, or that *the points* a *and* b *are separated by the set* A if a and b lay in different components of $\complement A$. Please note, that if a set A does not separate points a and b, then also none of its subsets do. Conversely, if a set separetes two points, then also all of its supersets not containing the said points separate them as well.

[2] The original proof was apparently divided to three papers, two by Brouwer and one by Lebesgue. A single article with the proof was not written at the time, propably due a controversy between Brouwer and Lebesgue about the definition of dimension. For further information about the history of the proof, see [Di2]. For a proof, see example `Find a good source`.

[3] Very illuminating descriptions of the Alexander horned sphere can be found from the internet, and I suggest to take a look for example in Wikipedia for pretty pictures.

Curves

Definition. By a *path* we mean a continuous mapping from the set $[0, 1]$ to the target space in question. We use the term *curve* as synonym for path. By a *loop* we mean a continuous mapping from the unit circle to the target space in question.

In this essay we will use the common abuse of notation, by referring by the word "path", "curve" or "loop" both to the mapping in question and its image. If a danger of confusion exists, we will emphasize the meaning by using the terms "image of the path/curve/loop γ" and "the function γ", respectively. When explicitly needed, we denote the image of a path γ by $|\gamma|$.

Definition. *A simple arc* is an embedding (i.e. an injective and continuous mapping) of the unit interval to the complex plane.

Definition. *A Jordan curve* is an embedding (i.e. an injective and continuous mapping) of the unit sphere to the complex plane. In this essay we often note a Jordan curve by c.

We denote the closed line segment between points a and b in the plane by $\overrightarrow{ab}$. More spesifically,

$$\overrightarrow{ab} = \{tb + (1-t)a \mid t \in [0, 1]\}.$$

When needed, we will interpret this as the constant speed path from a to b, i.e. as the mapping $t \mapsto tb + (1-t)a$. Please note that we will abuse notation also in this context, and refer with $\overrightarrow{ab}$ both to the path and the set.

Moreover if γ is a simple arc and $a, b \in \gamma$, then we denote the segment of the path γ from a to b by $\widetilde{ab}$. (Again, if we want to consider $\widetilde{ab}$ as a path instead of a set, we give it the induced parametrization of γ.)

The composition of two paths is denoted with a plus-sign. More spesifically, if γ_1 and γ_2 are paths defined on the unit interval I, then

$$\gamma_1 + \gamma_2 \colon I \to \Omega, \quad (\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(2t), & \text{when } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1), & \text{when } t \in [\frac{1}{2}, 1]. \end{cases}$$

The inverse of a path $\gamma \colon [0, 1] \to \mathbb{C}$ is the path

$$\overleftarrow{\gamma}(t) \colon [0, 1] \to \mathbb{C}, \quad \overleftarrow{\gamma}(t) = \gamma(1-t).$$

Definitions concerning function theory

Definition. We say that a domain Ω is *connected along the boundary* if for any point $z_0 \in \partial\Omega$ there exists arbitrarily small neighbourhoods U of z_0 such that $U \cap \Omega$ is connected.

This is a slightly different formulation of connectedness along the boundary than is given in the lecture notes of our course, but we have proven these two definitions to be equivalent in the exercises during our course.

Algebraic topology

Very little algebraic topology is needed during this essay. Or so it seems. We actually use in both our proofs results very similar to some essential results of algebraic topology.If you are not familiar with the subject, you may disregard the notions given here, but as I am a(n) (algebraic) topologist in my heart, I wish to make a few remarks. One reason being that those with prequisites in algebraic topology might acquire more insight to the events within the proof.

Also it is my firm belief that anyone interested in the studies of complex analysis would gain considerable benefit from studying even just the basics of homotopy theory, so I like to note every time algebraic topology is used in disguise to advertise its usefulness. The theorems and ideas referenced below can be found for example from [Hat].

First of all, any use of winding numbers is just talking about path homotopy with a different name[4]. More spesific uses of algebraic topology occur at two points. First being the Janizewski's theorem (theorem 2.3) used in the elementary approach to the proof. Janizewski's theorem can be seen as a modified version of the so called Seifert-van Kampen theorem of homotopy theory. This reflects to the fact that in our proof we use winding numbers.

The second direct use is (in some sense) in the proof of Brouwer fixed point theorem (theorem 2.8). This theorem was first proved without algebraic topology, and very fluent analytic proofs do exist for the Brouwer fixed point theorem, but the application of algebraic topology is in some sense very natural to use in the proof of this and 'similar' theorems. Also the most famous proofs of the claim (from my perspective) rely on algebraic topology. The proof in this essay is also done via the relation of homotopic paths and their respective integrals over analytic functions. But in this proof we have also some "higher" homotopy theory, for what we are saying in forming the contradiction could be rephrased to express the fact that S^1 is not a deformation retract of the closed disk $\overline{B}^2$. (i.e. there exists no continuous mapping from $\overline{B}^2$ to S^1 such that its restriction to the boundary would be the identical map).

1.3 How to prove the JCT?

For our course we need and in this essay shall prove the JCT as stated in 1.1. As mentioned, there are various approaches one could take to prove this theorem. One approach to prove the above theorem would be to prove directly the more difficult Jordan-Schöenflies theorem mentioned earlier, for the JCT would follow from this immidiately. We will not, however use this approach, as the Jordan-Schöenflies theorem can be obtained from the JCT[5].

I also hope that going through a proof that concentrates directly to the JCT one might better grasp the fundamental ideas of why the theorem holds. We shall prove the theorem with two different approaches.

[4]Or path homotopy is just defining winding numbers in a more general setting, as you please.

[5]Namely, in [So] the Jordan-Jordan-Schöenflies theorem is proved by applying an improved version of the Carathodory theorem concernig continuation of conformal mappings to the boundary. This version of the theorem requires the JCT.

The first one of them is “elementary”, which means that it is quite constructive and that the algebraic topology used is very well hidden. In this proof the basic reasons why the theorem holds are, I think, quite visible.

The other approach is less constructive and uses the Brouwer fixed point theorem. It is a bit less constructive, and the algebraic topology is somewhat more visible.

2 Proof of the Jordan curve theorem

We shall begin this section with a few lemmas that are used in both of our proofs.

Lemma 2.1. *Taken any path* $\gamma\colon [0,1] \to \mathbb{C}$ *(or any loop* $\gamma\colon S^1 \to \mathbb{C}$*) to the plane, the components of the complement of* γ *are open and path-connected.*

Proof. Unit interval and -sphere are compact and so is their respective images under a continuous mapping. Compact sets are closed in the plane and their complements are thus open by definition. Any component is by definition connected, and as a component of an open set also open. Open connected set of the plane are also path-connected, which gives the claim. □

Lemma 2.2. *Given a Jordan curve* J*, exactly one of the components of* $\mathbb{C} \setminus J$ *is unbounded. Clearly as the set* J *is compact, the set* $\mathbb{C} \setminus J$ *has at least one component.*

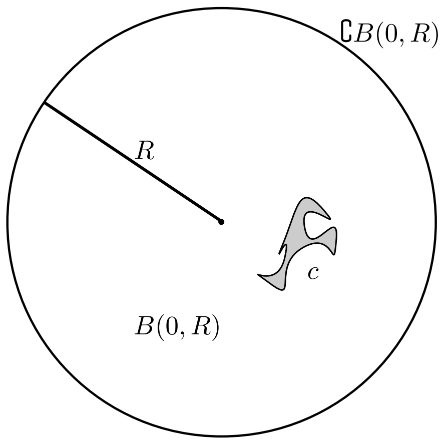

Figure 1: The complement of a Jordan curve has at most one boundary.

Proof. As a continuous image of a compact set the set J is compact and as such especially bounded. Thus there exists $r > 0$ such that

$$J \subset \overline{\mathrm{B}}(0, r) =: B_r.$$

(See picture 1.) The connected open set $\complement B_r$ is disjoint from J and so it has to be contained in one of the components of $\mathbb{C} \setminus J$, say U. Now any unbounded set, especially any unbounded component of $\mathbb{C} \setminus J$ intersects the set $\complement B_r$ and thus also the set U. A component of $\mathbb{C} \setminus J$ cannot intersect U without being U, which proves the first claim.

□

2.1 Proof by elementary means

We shall begin this subsection with a few auxiliary results. The proofs are from [Po].

The following result turns out to be very useful.

Theorem 2.3. *(Janiszewski's theorem) Let A and B be two compact subsets of the complex plane $\mathbb{C}$ and $a, b \in \mathbb{C} \setminus (A \cup B)$, $a \neq b$. If neither A nor B separates a and b and if $A \cap B$ is connected, then $A \cup B$ does not separate a and b.*

We prove the claim as a special case of the following lemma.

Lemma 2.4. *Let A_1 and A_2 be two closed subsets of the extended complex plane and $a, b \in \overline{\mathbb{C}}$. If neither A_1 nor A_2 separates a and b and if $A_1 \cap A_2$ is connected, then $A_1 \cup A_2$ does not separate a and b.*

Proof. We may assume that the said points are 0 and ∞.

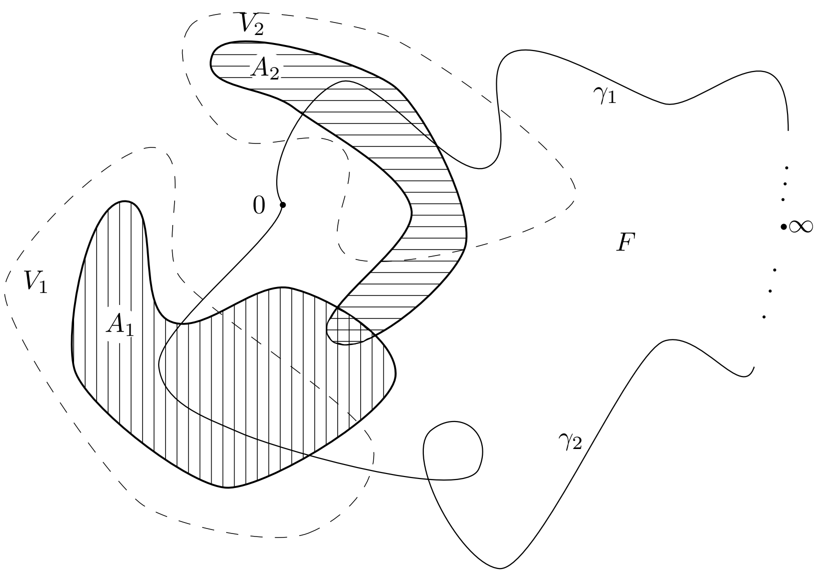

Figure 2: Constructing a branch of logarithm on a domain in the proof of Janizewski's theorem. (Theorem 2.3.)

For $i = 1, 2$ we may deduce the following:

As the set A_i does not separate the points 0 and ∞, we can find a path γ_i connecting these points such that $|\gamma_i| \cap A_i = \emptyset$. Let us look at the loop $\gamma_1 + \overleftarrow{\gamma_2}$. Note that it does not meet the connected set $A_1 \cap A_2$. If $A_1 \cap A_2 \neq \emptyset$, pick the component of $\overline{\mathbb{C}} \setminus |\gamma_1 + \overleftarrow{\gamma_2}|$ containing $A_1 \cap A_2$. If $A_1 \cap A_2 = \emptyset$, pick any component of the complement of this loop. In either case call the component F.

We have shown in function theory 2, that if a domain $\emptyset \neq \Omega \subsetneq \mathbb{C}$ is simply connected, we can for any $a \in \complement\Omega$ pick a well-defined branch of the function $z \mapsto \log(z - a)$. We have in FT2 also shown that a domain $\Omega \subset \mathbb{C}$ is simply connected exactly when $\overline{\mathbb{C}} \setminus \Omega$ is connected. Thus by these results we can pick a well-defined branch, f_i, of the logarithm on the set $\overline{\mathbb{C}} \setminus |\gamma_i|$, for the complement of each of these sets with respect to the extended complex plane is exactly the path-connected set $|\gamma_i|$ which contains the origin. Especially we may, and will, choose these branches such that they coincide in the set F. (Look at picture 2.)

Removing an open set from a compact set does not effect compactness, so the sets $A_i \setminus F, i = 1, 2$ are compact. Now the sets $A_i \setminus F$ and $|\gamma_i|$ were compact and disjoint, so there exists a strictly positive distance $r_i = d(A_i \setminus F, |gamma_i|)$. Thus we can find an open neighbourhood V_i of $A_i \setminus F$ (for example $B(A_i \setminus F, r_i)$) such that V_i does not intersect the path γ_i. The sets $A_1 \setminus F$ and $A_2 \setminus F$ are also disjoint compact sets so we may require that the sets V_i are disjoint.

Denote $H = V_1 \cup V_2 \cup F$. (Look harder at picture 2.) We can define the function

$$f\colon H \to \mathbb{C}, \quad f(z) = \begin{cases} f_1(z), & \text{when } z \in V_1 \\ f_2(z), & \text{when } z \in V_2 \\ f_1(z) = f_2(z), & \text{when } z \in F \end{cases}$$

The function f is analytic in the domain $H \supset A_1 \cup A_2$ and satisfies for any $z \in H$ the relation $\exp f(z) = z$.

We are now ready to show that the set $A_1 \cup A_2$ does not separate the points 0 and ∞. Assume the contrary. This would imply, that also the set H separates these points. This means that the point 0 must lie in a bounded component of the complement of H.

By imitating We may now construct a path γ whose image lies in H such that winding number $\eta(\gamma; 0)$ of γ with respect to 0 is 1. This construction can be found from several sources, for example from:

- In [FT2, p. 84] the proof of theorem 5.19 contains this construction. (The closed sets A and B in the proofs of this and the following source correspond in our proof to the bounded component of $\complement H$ containing 0, and the unbounded component of $\complement H$ containing the point ∞, respectively.)
- In [Ah, p. 139] the proof of theorem 14 contains this construction.
- In [Ru, p. 274] the theorem 13.11 implies the existence of such a path.

But this is a contradiction, as

$$\eta(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma z^{-1}\, \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \underbrace{f'(z)}_{\text{analytic}}\, \mathrm{d}z = 0 \neq 1.$$

Thus the set $A_1 \cup A_2$ cannot separate the points 0 and ∞. ☐

With this we can easily prove Janizewski's theorem.

Proof of Janizewski's theorem, theorem 2.3. Compact sets of the complex plane are closed in the extended complex plane, so by applying lemma 2.4 we get our claim. ☐

From Janizewski's theorem we can prove the following useful corollary.

Corollary 2.5. *The complement of a simple arc is connected (i.e. it does not separate any points not contained in its image).*

Proof. Assume

$$\gamma\colon [0,1] \to \mathbb{C}$$

is a simple arc. Take any $a, b \in \mathbb{C} \setminus |\gamma|$. Because the image of the path γ is compact, we have positive distances $d(a, |\gamma|) > 0$ and $d(b, |\gamma|) > 0$. Take r to be the smaller of these two.

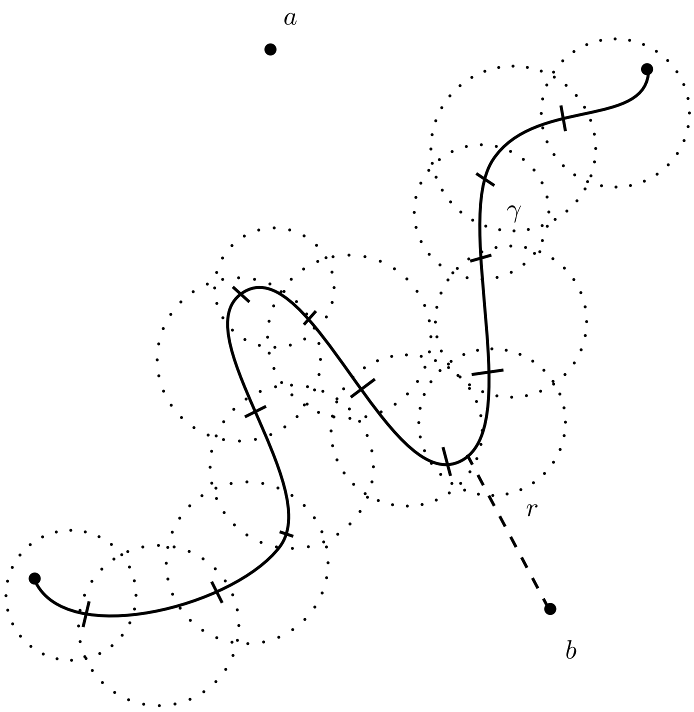

Figure 3: The complement of a simple arc is connected.

We can now by compactness of $[0,1]$ partition the interval $[0,1]$ into n closed subintervals

$$[0, t_1], [t_1, t_2], \ldots, [t_{n-1}, 1]$$

such that each restriction $\gamma_i := \gamma|_{[t_i, t_{i+1}]}$ is contained in an open set with diameter less than r. Now each of the restrictions γ_i is especially contained in a ball B_i not containing either of the points a or b. As we can connect the points a and b in the complement of any of these balls B_i, we can also connect them in a larger set $\mathbb{C} \setminus |\gamma_i|$. This means that none of the sets $|\gamma_i|$ separate the points a and b. (Look at picture 3.)

We can now prove the claim by induction over the indeces of our partition of the unit interval.
Claim: The simple arc $\gamma_1 + \ldots + \gamma_i$ does not separate points a and b for any $i = 1, \ldots, n$.

Base case: As we noted earlier, none of the paths γ_i separete the said points, so especially this holds for the path γ_1.

The inductive step: Assume that the path

$$\alpha := \gamma_1 + \ldots + \gamma_n$$

does not separate the points a and b. Now we must have that $\alpha \cap \gamma_{n+1} = \{\gamma_n(t_{n+1})\}$, for otherwise the path γ would not be injective, contrary to what we assumed. Now by assumption the path α does not separate points a and b, and neither does the path γ_{n+1} by the previous notion. Also their intersection is just a singleton, which is clearly a connected set. But this means that by Janizewski's theorem the union

$$|\alpha| \cup |\gamma_{n+1}| = |\gamma_1 + \ldots + \gamma_n + \gamma_{n+1}|$$

does not separate those points and this is exactly what we wanted to show.

Thus the claim holds. □

Lemma 2.6. *Given a Jordan curve J the boundary of each component of $\mathbb{C} \setminus J$ is exactly J.*

Proof. Let A be a connected component of $\mathbb{C} \setminus J$. We will prove that $\partial A = J$ "one direction at a time".

"$\subset$": (Note, that for this direction we would only need to know that J is closed.) Let $z_0 \in \partial A$. Because A is open, we must have that $z_0 \notin A$. On the other hand the complement of A consists of J and (possibly) the other components of $\mathbb{C} \setminus J$. If z_0 were to belong to some other component of $\mathbb{C} \setminus J$, then we would find a neighbourhood of the point z_0 contained in this component. But then we would get a contradiction, as we would by definition of boundary find points of the set A within this neighbourhood. The components would then have common points an would then necessarily be same. Thus we must have $z_0 \in J$, as we wanted.

"$\supset$": Let $z_0 \in J$. It suffices to show that $z_0 \in \overline{A}$, so we only need to find points in A arbitrarily close to z_0. Look at picture 4 during the following.

Take a point $x \in A$. Take $n \in \mathbb{N}$ so large, that

$$\frac{1}{n} < \min(d(z_0, x), d(|J|)),$$

and denote

$$D_n =: B\big(z_0, \frac{1}{n}\big).$$

Call by J' the component of $D_n \cap J$ containing the point z_0. The mapping J is a homeomorphism between the sets $|J|$ and S^1, so the pre-image of the set $|J'|$ under the mapping J is also necessarily an open connected subset of S^1. Thus the complement of this pre-image is homeormorphic to the closed unit interval and gives rise to a simple arc as a restriction of the mapping J to this complement in question. Especially the image of this arc then equals the complement of J' in J, $J'' := J \setminus J'$. By corollary 2.5 this simple arc thus does not separate the points x and z_0. As open components of $\mathbb{C} \setminus J''$ are necessarily path connected by our previous notion, this means that we can find a curve

$$\gamma_n \colon [0,1] \to \mathbb{C} \setminus J'' \text{ such that } \gamma_n(0) = x \text{ and } \gamma_n(1) = z_0.$$

This curve connects the points x and z_0 in $\mathbb{C} \setminus J''$. Set

$$t_0 = \inf \underbrace{\{t \in [0,1] \mid \gamma_n(t) \in \partial D_n\}}_{=:B}, \quad z_n = \gamma_n(t_0).$$

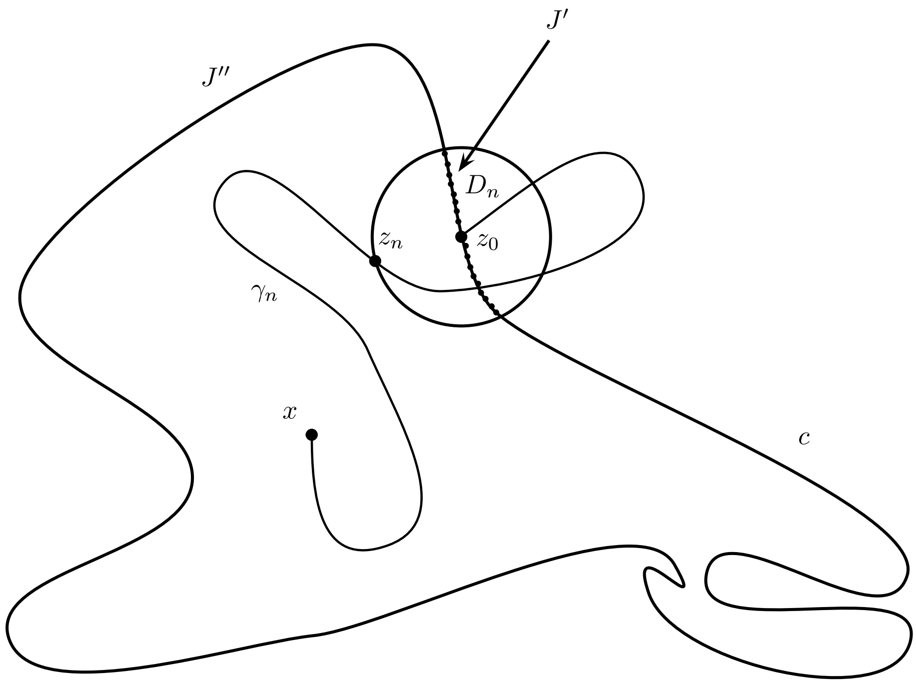

Figure 4: Finding points of a component arbitrarily close to the Jordan curve.

The set B is bounded from below by definition. It is also non-empty, for we note that the mapping $t \mapsto d(\gamma_n(t), z_0)$ is continuos as a composition of two continuos functions, and we have that

$$d(\gamma_n(1), z_0) = d(z_0, z_0) = 0 < \frac{1}{n} < d(x, z_0) = d(\gamma_n(0), z_0),$$

because we chose n to be so large that the second inequality holds. Thus by Bolzano theorem we must have a point $t \in [0, 1]$ such that $d(\gamma_n(t), z_0) = 1/n$, i.e. $\gamma_n(t) \in \partial D_n$.

The point z_n is the first point where the path γ meets the sphere ∂D_n. As the image of the arc $\gamma_n|_{[0,t_0]}$ intersects neither J'' (by definition), nor J' (if it would intersect the set J', we would get a contradiction by to the definition of an infimum by applying the Bolzano theorem to the earlier function with endpoints 0 and t_0), we have that $\gamma_n|_{[0,t_0]} \subset \mathbb{C} \setminus J$. But now we note that the path $\gamma_n|_{[0,t_0]}$ has to lie within the component A, for it is contained in the complement of the curve J, and as a path all its points must lie in the same path component as x, which is exactly the component A.

So $\gamma_n|_{[0,t_0]} \subset A$, and thus $z_n \in A$. Now as $n \to \infty$, these points converge towards z_0, and we have thus shown that $z_0 \in \bar{A}$.

Thus we know that J is the boundary for every component of its complement. $\square$

Remark 2.1. After this point one might want to jump to hasty conclusions claiming that the JCT clearly follows from the previous lemma, as we have a curve with clearly just two sides and by the previous lemma all components must "touch" this two-sided curve at each point. This lemma *does* play a crucial role in our proof, but to give some aspects of the nontriviality of the remaining part of the proof, I would like to note that one can construct[6] something called "Lakes of Wada", which are three disjoint domains in the plane with a common boundary. So we need to find a formal way to get our hands on the "two-sidedness" of J.

What is left to show is the following lemma.

Lemma 2.7. *The complement of a Jordan curve has exactly two components.*

Proof. We will imitate the proof given in [Di1, p. 256] and prove the JCT in two cases:

Case 1. Assume the image of the Jordan curve *contains a line segment*.

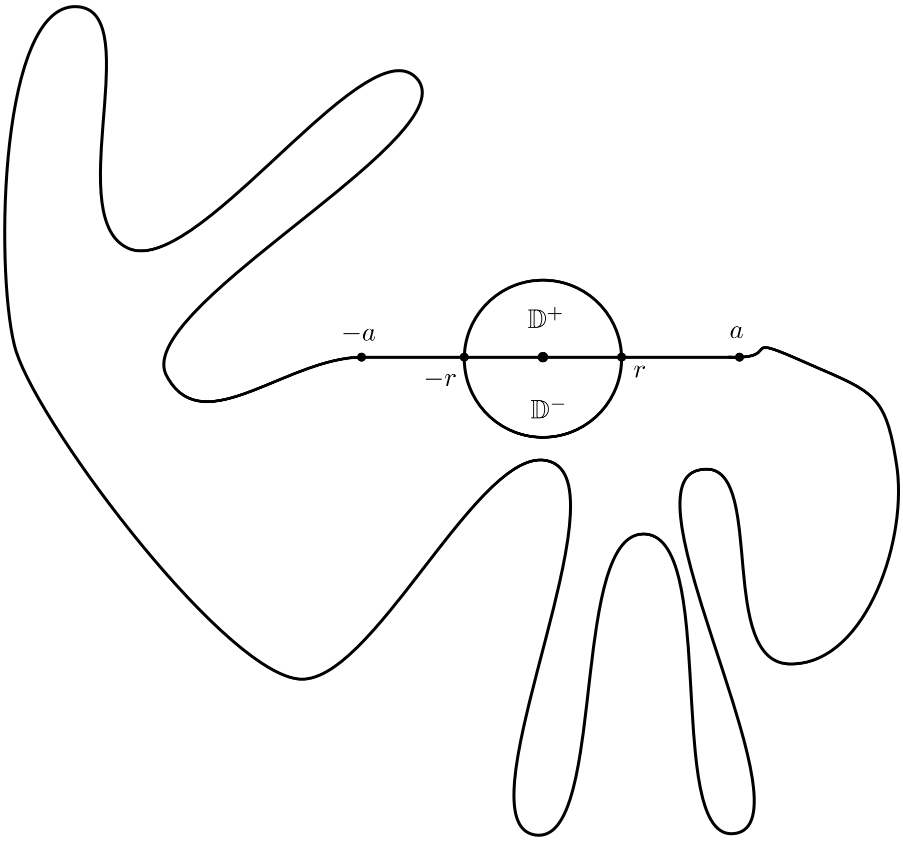

Figure 5: Proof of the JCT when the curve contains a linesegment.

[6]This can be found formally from [Yo], but a description and pictures can be found for example from Wikipedia.

Denote the assumed line segment by I. Translations and rotations are homeomorphisms of the whole plane, so we may assume that $I = [-a, a]$, where $a \in \mathbb{R}, a > 0$.

One easily chects that the set $J \setminus \operatorname{int} I$ is compact. Thus we have that there exists a strictly positive distance $d(0, J \setminus \operatorname{int} I)$. Set $r = \frac{1}{2} d(0, J \setminus \operatorname{int} I)$ and denote $\mathbb{D}_r = \mathrm{B}(0, r)$. Now $\mathbb{D}_r \cap J = [-r, r]$. Denote

$$\mathbb{D}^+ = \{z \in \mathrm{B}(0,r) \mid \operatorname{Im} z > 0\}\,, \quad \mathbb{D}^- = \{z \in \mathrm{B}(0,r) \mid \operatorname{Im} z < 0\}\,.$$

Now as $0 \in J$, we have by Lemma 2.6 that 0 lies in the boundary of every component of the set $\mathbb{C} \setminus J$. That means that each of the components must have common points with either $\mathbb{D}^+$ or $\mathbb{D}^-$. But as these are both connected as convex sets and are contained in $\mathbb{C} \setminus J$, we see that each component of the set $\mathbb{C} \setminus J$ must contain either $\mathbb{D}^+$ or $\mathbb{D}^-$.

Thus we have shown that $\mathbb{C} \setminus J$ *has at most two components.* To complete the proof we only need to show that the set $\mathbb{C} \setminus J$ is not connected.

We will do this by showing that any points $x \in \mathbb{D}^+$ and $y \in \mathbb{D}^-$ belong to different components. We will proceed with a counter-assumption followed by an application of Janiszewski's theorem (Theorem 2.3).

Assume that the set $\mathbb{C} \setminus J$ is connected. Pick points $x \in \mathbb{D}_+$, $y \in \mathbb{D}_-$. As the set J is compact, we find $R > 0$ so large that $J \subset B(0, R)$. We note that clearly the set $G := \complement \mathbb{D}_r \cap \overline{B}(0, R)$ does not separate points x and y, as the set $\mathbb{D}_r$ is connected as a convex set. By our counterassumption also the set J does not separate the points x and y. But now the intersection of the sets J and G is the complement of the interval $]-r, r[$ in J, so it is a simple arc. Thus by corollary 2.5 it does not separate the points x and y, and by Janizewski theorem neither does the union $G \cup J$. But this is a contradiction, as the complement of this set is just

$$\mathbb{D}_+ \cup \mathbb{D}_- \cup \complement \overline{B}(0, R) \supset \mathbb{D}_+ \cup \mathbb{D}_-,$$

which clearly separates the points x and y.

The antithesis was false, so we see that the set $\mathbb{C} \setminus J$ is not connected. Combining this with the previous result we see that the set $\mathbb{C} \setminus J$ has exactly two components.

Case 2. Assume the image of the Jordan curve *does not contain a line segment.*

Let $a, b \in J$, $a \neq b$. Denote the line segment $\overrightarrow{ab}$ in the plane between a and b by I. As before, we may assume that $I = [-a, a]$, where $a \in \mathbb{R}$, $a > 0$. By our assumption, there exists at least on interior point, say x_0 in I that does not lie on the image of the curve J. (See picture 6.) By our previous notions we know that the set $\complement J$ is open, and thus x_0 lies in the set $(\operatorname{int} I) \cap \complement J$ which is an open subset of $\operatorname{int} I$ with respect to the topology induced by the plane. Thus there exists a ball in I, which in this case is an interval, $I_0 :=]x_0 - \delta, x_0 + \delta[$ such that $I_0 \cap J = \emptyset$.

Let us now find the "largest" interval around x_0 such that it is still contained in the complement of the set J. More spesifically, set

$$\alpha = \inf\{x \in I \mid]x, x_0] \subset \complement J\}$$

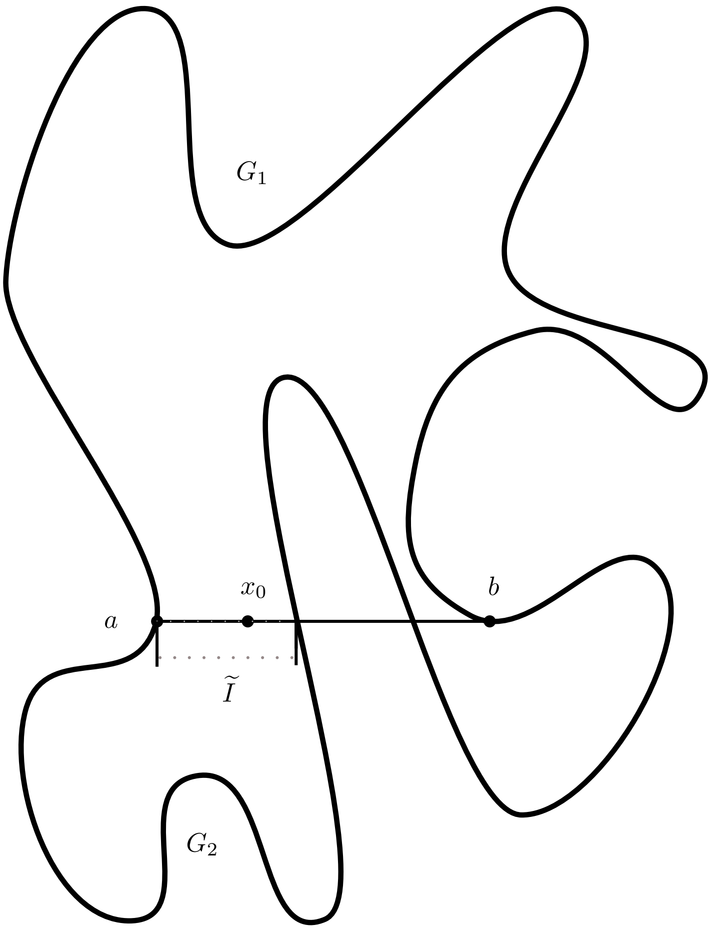

Figure 6: Finding a segment not meeting the Jordan curve.

and in a similar fashion

$$\beta = \sup\{x \in I \mid [x_0, x[\subset \complement J\}.$$

Both α and β are well-defined, for the respective sets are nonempty by the earlier notion and bounded by $\pm a$.

Now set $\widetilde{I} =]\alpha, \beta[$. Next we wish to divide the curve J into two parts by using $\widetilde{I}$ so that we can apply the result of Case 1. Note that $J \setminus \{\alpha, \beta\}$ consists of two simple arcs. Let us call them G_1 and G_2.

What we do is that we will form two new jordan curves, c_1 and c_2, (see picture) such that:

c_1 follows first G_1 and then goes through $\widetilde{I}$.

c_2 first goes through $\widetilde{I}$ (in a different direction as c_1) and then follows G_2.

I suggest to look at the picture 7 at this point.

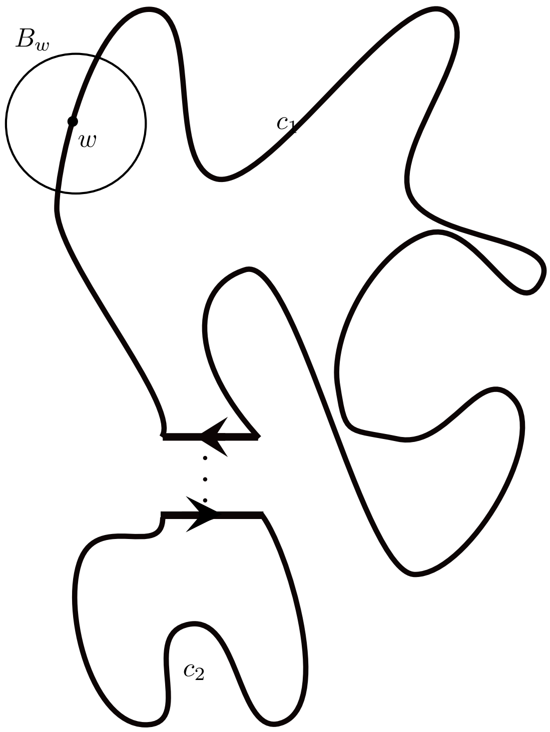

Figure 7: Constructing new Jordan curves with a segment.

Let us do this in rigour: Reparametrize[7] $J\colon S^1 \to \mathbb{C}$ such that $J(1) = \alpha, J(-1) = \beta$. Now denote by L_1 the affine bijection from $[0,\pi]$ to $[\beta,\alpha]$, by L_2 the affine bijection from $[\pi, 2\pi]$ to $[\alpha,\beta]$ and define (in the following we denote $\arg(z) =: t$)

$$c_1\colon S^1 \to \mathbb{C}, \quad c_1(e^{it}) = \begin{cases} J(e^{it}), & \text{when } t \in [0,\pi] \\ L_2(t), & \text{when } t \in [\pi, 2\pi], \end{cases}$$

$$c_2\colon S^1 \to \mathbb{C}, \quad c_2(e^{it}) = \begin{cases} L_1(t), & \text{when } t \in [0,\pi] \\ J(e^{it}), & \text{when } t \in [\pi, 2\pi]. \end{cases}$$

Both c_1 and c_2 are seen to be Jordan curves, which both contain a line segment $\widetilde{I}$. Now we are ready to apply the result from case 1.

Take a point $w \in |c_1| \setminus \overrightarrow{\alpha\beta}$. The point w is not contained in the compact set $|c_2|$ and thus there is exists an open ball B_w containing the point w such that $B_w \cap |c_2| = \emptyset$. By the result of case 1, there are exactly two components of $\mathbb{C}\setminus|c_1|$, and by lemma 2.6 both of the components intersect the ball B_w.

[7] This can be done, as reparametrization does not effect the continuity of J and the continuity is the only property of J that we need here.

Now if we take points $w', w'' \in B_w$ that are in the same component of $\mathbb{C} \setminus |c_1|$, they are not (of course) separated by the set $|c_1|$. Neither are they separated by the set $|c_2|$ for they belong to a connected subset of $\mathbb{C} \setminus |c_2|$, namely the ball B_w. Also by corollary 2.5 the simple arc $\overrightarrow{\alpha\beta}$ that is the intersection of $|c_1|$ and $|c_2|$ is connected and thus by Janizewski's theorem they are not separated by the set $|c_1| \cup |c_2|$. Especially they are not separated by the smaller set $J \subset |c_1| \cup |c_2|$. This means that the points w' and w'' belong to the same connected component of $\mathbb{C} \setminus J$.

The conclusion is, that by lemma 2.6 every component of $\mathbb{C} \setminus J$ has points in B_w, and now by Janiszewski's theorem every pair of points in B_w that is not separated by the set $|c_1|$ is also not separated by the set J. Thus the set $\mathbb{C} \setminus J$ must have *at most* as many components as the set $\mathbb{C} \setminus |c_1|$. But by the result of case 1 the set $\mathbb{C} \setminus J$ has exactly two components. Thus *the set* $\mathbb{C} \setminus J$ *has at most two components.*

What is left to show is that the set $\mathbb{C} \setminus J$ is not connected. By the results of case 1, we know that there exists two points $w', w'' \in B_w$ such that the set $|c_1|$ separates them. We wish to show that they are separated by the set J as well.

Assume that the points w' and w'' are not separated by the set J. As before, we would have that they are not separeted by $|c_2|$ (again, they lie in a connected set $B_w \subset \complement |c_2|$) nor by the simple arc which is the intersection of $|c_2|$ and J. Then by Janizewski's theorem they are not separated by the set $J \cup |c_2|$. But this gives a contradiction, as then they would not be separated by $|c_1| \subset J \cup |c_2|$.

We conclude that the set $\mathbb{C} \setminus J$ has exactly two components.

$\square$

The JCT is now proven with elementary methods. Let us move on to the proof that involves the Brouwer fixed point theorem.

2.2 Proof via Brouwer's fixed point theorem

This approach to the proof is from the lecture notes [GC], which in turn was based on the article [Ma]. We will use in this section terminology of linear algebra rather than complex analysis, because the proofs are in nature more elementary plane geometry than complex analysis. An exception will be the following proof of Brouwer fixed point theorem, for in the proof we well need winding numbers which are most naturally expressed via complex notation.

Theorem 2.8. *(Brouwer fixed-point theorem) Given a continuous function f from the closed unit disk $\overline{\mathbb{D}}$ to itself, there exists a point $z_0 \in \overline{\mathbb{D}}$ such that $f(z_0) = z_0$. Such a point is called a fixed point of the function f and is not unique in general.*

Proof. Let us make a counter-assumption that for some continuous function $f\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ there exists no fixed point. Now we can define a continuous mapping

$$g\colon \overline{\mathbb{D}} \to S^1,$$

which has the property $g|_{S^1} = \mathrm{id}$.

Geometrically this function is constructed as follows: For each $z \in \overline{\mathbb{D}}$ consider the line segment starting from $f(z)$ and going through z. Find the first point at which this line intersects the unit sphere and call this point $g(z)$.

Formally we can define

$$g(z) = z + a\frac{z - f(z)}{|z - f(z)|}, \text{ where } a = -\operatorname{Re} b\overline{z} \pm \sqrt{(\operatorname{Re} b\overline{z})^2 + 1 - |z|^2}$$
$$\text{and } b = \frac{z - f(z)}{|z - f(z)|}.$$

From this formal definition we see the function g to be continuos as a composition of continuous functions.

Please note that both of these definitions of g require the fact that $f(z) \neq z$ for all $z \in \overline{\mathbb{D}}$.

Now let us look at the loop

$$\gamma\colon [0,1] \to S^1, \gamma(t) = \mathrm{e}^{it}.$$

We note that as $g|_{S^1} = \mathrm{id}$, the path formed as a pre-image of γ is also a path defined by $z \mapsto \mathrm{e}^{it}$, but in $\overline{\mathbb{D}}$. As the closed ball is simply connected, there exists a homotopy

$$H\colon [0,1] \times [0,1] \to \overline{\mathbb{D}}$$

that deforms γ to a constant path in $\overline{\mathbb{D}}$. Now the mapping $g \circ H$ gives a homotopy in S^1 that also deforms γ into a constant path γ_c in S^1.

This is a contradiction, since we know by for example Function Theory II, that the winding number $\eta(\gamma; 0)$ of the path γ in S^1 is 1, but as the integral of an analytic function over continuous paths is a homotopy invariant, we must have

$$\eta(\gamma; 0) = \frac{1}{2\pi i}\int_\gamma z^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\gamma_c} z^{-1}\,\mathrm{d}z = 0 \neq 1.$$

Thus every continuous function from the unit disc to itself must have a fixed point. $\square$

We next prove a corollary to the Brouwer fixed point theorem that will be used extensively throughout the proof of the JCT.

Corollary 2.9. *Let $I = [-1, 1]$ and assume that*

$$h\colon I \to I^2, \quad h(t) = (h_1(t), h_2(t)) \text{ and}$$
$$v\colon I \to I^2, \quad v(t) = (v_1(t), v_2(t))$$

are two paths such that[8]

$$h_1(-1) = -1, h_1(1) = 1, v_2(-1) = -1 \text{ and } v_2(1) = 1.$$

Then for some $t, s \in I$ we have that $h(t) = v(s)$.

Proof. Let us first note, that as the unit rectangle is homeomorphic to the unit disk, the Brouwer fixed point theorem holds also for the unit rectangle, i.e. every continuous map

$$f\colon I^2 \to I^2$$

has a fixed point.

Let us make a counterassumption; $h(t) \neq v(s)$ for all pairs of points $t, s \in I$. Define a 'maximum of distances':

$$N(s,t) = \max\left(|v_1(t) - h_1(s)|, |v_2(t) - h_2(s)|\right)$$

and a continuous function

$$f\colon I^2 \to I^2, \quad f(s,t) = (\underbrace{\frac{v_1(t) - h_1(s)}{N(s,t)}}_{=:f_1(s,t)}, \underbrace{\frac{h_2(s) - v_2(t)}{N(s,t)}}_{=:f_1(s,t)}).$$

By the Brouwer fixed point theorem there exists a fixed point, say (s_0, t_0), of this function.

Now by definition we have that $f[I^2] \subset \partial I^2$.

Thus we must have either that $s_0 = \pm 1$ or $t_0 = \pm 1$. All of these cases create a contradiction in a very similar fashion. We shall go through the case $t_0 = 1$. If this were true, we would have that $(s_0, 1) = f(s_0, 1)$, so especially the second coordinates would agree. So we would have that

$$1 = f_2(s_0, 1) = \frac{h_2(s_0) - v_2(1)}{N(s_0, 1)} \leq \frac{1 - v_2(1)}{N(s_0, 1)} = \frac{1 - 1}{N(s_0, 1)} = 0$$

which is a contradiction. □

We will continue by proving a modified version of lemma 2.6. This modified version states that the image of the Jordan curve is the boundary for each of the components of its complement given the assumption that there exists at least two components in the said complement.

Main motivation of this is to keep the proofs as self-contained and compact as possible. More specifically; in the elementary proof we needed Janizewski's

[8](Geometrically this condition means that the path h starts from the bottom of the square and ends in the top, and the path v starts from the left side of the square and ends in the right.)

theorem (theorem 2.3), with which it was quite fluent to prove the lemma. In this proof we need only a weaker version, and the work to go through all the lemmas in the previous section just to get this simple result would be kind of an overkill. Especially when you consider how nicely this alternate proof fits to our 'Brouwerian' framework.

> "Never use a cannon to kill a fly."
> –Confucius

Before the actual proof, however, we need to prove the following lemma, which is (in its full generality) known as the Tietze mapping theorem. It is proven here in the special case of the plane. Statement and proof of the general case can be found for example from [Wi] (or from [Vä], from which the prove here is mimiced, if you can read finnish).

Lemma 2.10. *Let $A \subset \mathbb{R}^2$ be closed and $f\colon A \to [a,b]$ continuous. Then there exists a continuous mapping $g\colon \mathbb{R}^2 \to [a,b]$ such that $g|_A = F$.*

Proof. First of all, note that if we take any two closed disjoint subsets A_1 and A_2 of the plane, the function

$$h\colon \mathbb{R}^2 \to [0,1], \quad h(x) = \frac{d(x,A_1)}{d(x,A_1)+d(x,A_2)}$$

is well defined (because the sets were closed, no point can have a zero distance to both of them without belonging to both of them) and continuous as a composition of the continuous distance-mapping and a rational function with a non-vanishing denumerator. Also we see that $h[A_1] = \{0\}$, and $h[A_2] = \{1\}$. From this we see (e.g. by concidering partially linear transformations) that we can for any two closed disjoint subsets of the plane find a function that maps the plane continuously to any given closed interval of the real line such that the closed sets are mapped to two given distinct points within this interval.

Now let $f\colon A \to [a,b]$ be continuous. We may assume by studying an affine transformation that $a = -1$, $b = 1$. The sets $A_1 := f^{-1}[-1,-\frac{1}{3}]$ and $A_2 := f^{-1}[\frac{1}{3},1]$ are closed as pre-images of closed sets with respect to a closed mapping. Thus by our earlier notion we find a continuous function $h_1\colon \mathbb{R}^2 \to [-\frac{1}{3},\frac{1}{3}]$ such that $h_1[A_1] = \{-\frac{1}{3}\}$, and $h_1[A_2] = \{\frac{1}{3}\}$. Now we note, that actually for every $x \in A$ we have

$$|f(x) - h_1(x)| \leq \frac{2}{3}.$$

Now if we re-apply this procedure to the function $(f - h_1)\colon A \to \left[-\frac{2}{3},\frac{2}{3}\right]$ we find a continuous function $h_1\colon \mathbb{R}^2 \to [-\left(\frac{2}{3}\right)^2, \left(\frac{2}{3}\right)^2]$ such that

$$|f(x) - h_1(x) - h_2(x)| \leq \left(\frac{2}{3}\right)^2$$

for all $x \in A$. By induction, we thus find a sequence (h_n) of continuous functions defined on the whole plane such that

$$|f(x) - \sum_{n=1}^{k} h_n(x)| \leq \left(\frac{2}{3}\right)^k$$

for all $k \in \mathbb{N}$, $x \in A$. Also we see that

$$\sum_{n\in\mathbb{N}} |h_n| \leq \sum_{n\in\mathbb{N}} \left(\frac{1}{3}\right)^n < \infty.$$

Combining these two facts, we actually see that the function

$$g\colon \mathbb{R}^2 \to [-1,1], \quad g(x) = \sum_{n=1}^{\infty} h_n(x)$$

exists, is continuous and equals the function f in the set A. This proves the claim. $\square$

Now we are ready to continue.

Theorem 2.11. *If the complement of a given a Jordan curve J is not connected, then the boundary of each component of $\mathbb{R}^2 \setminus J$ is exactly J.*

We will prove this theorem as an corollary of the following lemma, which was also proven in the elementary proof. Again we shall prove it with tools more fitted to the Brouwerian framework.

Lemma 2.12. *A simple arc cannot separate any pair of points in the complement of its image.*

Proof. We will prove the claim by creating a contradiction to a counter-assumption with the Brouwer fixed point theorem, i.e. by constructing a continuous function from the closed unit disk to itself without a fixed point. So let us assume the contrary, i.e. that the complement of a simple arc γ is not connected.

One can easily check (for example, by modifying the proof of lemma 2.2) that the set $\mathbb{R}^2 \setminus \gamma$ has exactly one unbounded component. As we assumed that there exists at least two components, there must especially exist a bounded component A of the set $\mathbb{R}^2 \setminus \gamma$. Let $x_0 \in A$.

Choose $r > 0$ so large that $\gamma \subset B(x_0, r)$. This is possible as the set γ is bounded as a compact set. The boundary $S := \partial B(x_0, r)$ is now contained in the unbounded component of the set $\mathbb{R}^2 \setminus \gamma$.

Now by applying lemma 2.10 to the mapping

$$\mathrm{id} \circ \gamma^{-1}\colon |\gamma| \to [0,1]$$

we see that the identity mapping $\mathrm{id}\colon |\gamma| \to |\gamma|$ can be extended to a continuous mapping $s\colon \overline{B}(x_0, r) \to |\gamma|$.

Let us now define $q\colon \overline{B}(x_0, r) \to \overline{B}(x_0, r) \setminus \{x_0\}$ by setting

$$q(z) = \begin{cases} s(z), & \text{when } z \in \overline{A} \\ z, & \text{when } z \in \complement A \cap \overline{\mathrm{B}}(x_0, r). \end{cases}$$

The mapping is well defined and continuous, as both mappings r and id are, and their respective domains only meet in the set $\overline{A} \cap \complement A$, which is contained in the simple arc in which the mapping s equals the identity. Especially note that $q|_S = \mathrm{id}$. Also now the point x_0 does not belong to the image set of the mapping, because as we chose it from the set A and thus it is mapped via the mapping s, which has as its image the set $|\gamma|$ which does not contain the point x_0.

To simplify a definition, we assume that $x_0 = 0$. Let

$$p\colon \overline{B}(0,r) \setminus \{0\} \to S, \quad p(z) = \frac{z}{|z|}.$$

Now we actually see, that the composition of p and q form a continuos mapping from the closed unit disk to the unit sphere that keeps the unit circle fixed. This was seed to be impossible in the proof of the Brouwer fixed point theorem, so at this point one could redo parts of that proof and create a contradiction. But as we have the Brouwer fixed point theorem in our hands, we can save effort by applying it here.

So denote still

$$t\colon S \to S, \quad t(z) = -z.$$

and note that the mapping

$$t \circ p \circ q\colon \overline{B}(0,r) \to S \subset \overline{B}(0,r)$$

contradicts the Brouwer fixed point theorem, for the image set is contained in S, so any fixed point must lie here. But on the other hand we see that the mappings p and q restricted to the boundary are just identity mappings, so for any $z \in S$ we have that

$$(t \circ p \circ q)(z) = t(z) = -z \neq z.$$

□

We are now ready to prove our claim that the Jordan curve is the boundary of all the components of its complement, if there exists at least two of such components.

Proof of lemma 2.11. Let A be a connected component of $\mathbb{R}^2 \setminus J$. We will prove that $\partial A = J$ "one direction at a time".

"$\subset$": This follows from basic topology and is done in detail earlier in the proof of lemma 2.6

"$\supset$": Assume the contrary, that is, that $\partial A \subsetneq J$. Let us pick a point $a \in J \setminus \partial A$. As the set ∂A is a closed subset of a compact set J, it is necessarily compact. This means that there exists a neighbourhood U of the point $a \notin A$ such that $U \cap A = \emptyset$. So especially there exists a simple arc γ (one can take for example the component of $J \cap \complement U$ containing ∂A with induced parametrization from the curve J) that contains the set ∂A.

But now our previous theorem states that the simple curve γ cannot separate any two points. This means, that if we pick two points, $x \in A$, and y in some other component of $\mathbb{C} \setminus J$, we can take a path α connecting these two in $\complement(\partial A)$. But our assumption implies that the Jordan curve J separates these two points, so the path α must cross the set $J \setminus \gamma$. This is impossible, as then the first crossing point would belong to the boundary of A, but the crossing points belong to a set disjoint from ∂A.

Thus we know that if $\mathbb{R}^2 \setminus J$ is not connected, then J is the boundary for every component of its complement.

$\square$

(If you have not read the proof by elementary means, remark 2.1 might turn out to be a good thing to read at this point for motivation.)

Now all that is left to prove after our lemmas is that the complement of the Jordan curve has exactly one bounded component.

Proof of the JCT. We begin the proof by fixing some points on our curve. We will especially find a nice interior point of the complement of the image of our Jordan curve. We will apply our corollary 2.9 several times.

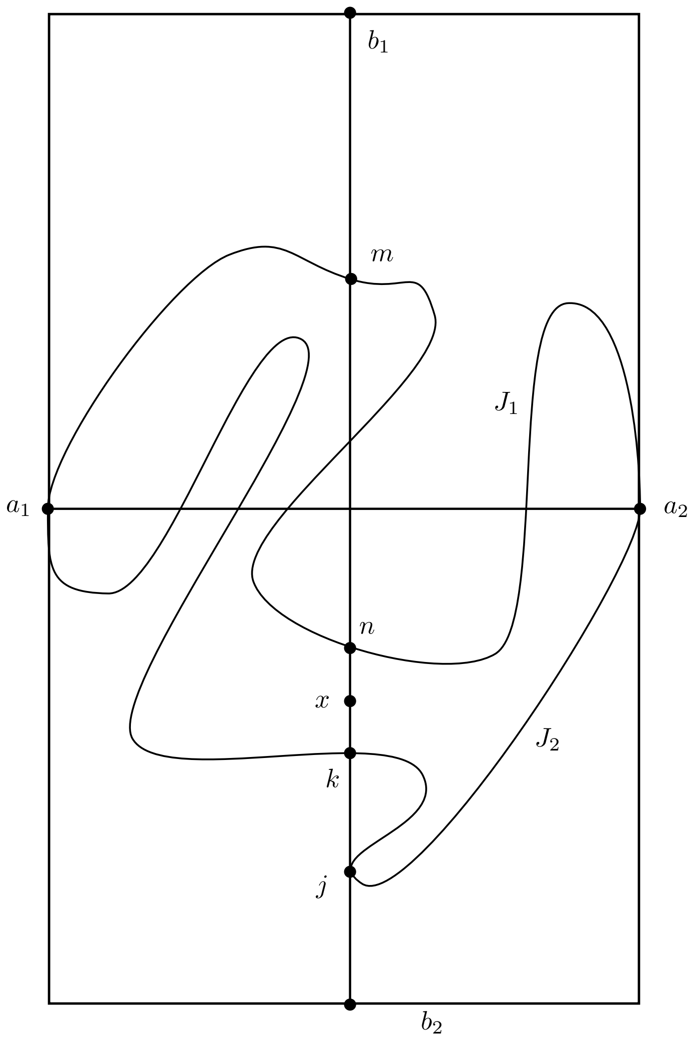

Figure 8: Fixing of points on the Jordan curve.

As the set J is compact, we can find points a_1 and a_2 in J such that $d(a_1, a_2) = d(J)$. We may again assume, that $a_1 = -1, a_2 = 1$. Now the image of the curve is contained in the rectangle $[-1,1] \times [-2,2]$ which we denote by R. Set $b_1 := (0,2)$ and $b_2 := (0,-2)$ (i.e. the midpoints of the top and bottom of the rectangle). Note that $R \cap J = \{a_1, a_2\}$, for otherwise the distance would not be maximal.

The segment $\overrightarrow{b_1b_2}$ meets the Jordan curve J by our lemma 2.9 (You can apply the lemma to any segment of J between the points a_1 and a_2). Pick $m \in J \cap \overrightarrow{b_1b_2}$ with the largest second coordinate (this is possible, because the intersection is a compact set).

Removing the points a_1 and a_2 from the Jordan curve divides it to two simple arcs). Let us denote the one that goes through the point m as J_1 and the other as J_2.

Pick (again by lemma 2.9 and compactness) element $n \in J_1 \cap \overrightarrow{b_1b_2}$ with the smallest second coordinate.

Now the paths J_2 and $\overrightarrow{nb_2}$ intersect, for otherwise the paths J_2 and $\overrightarrow{b_2n} + \widetilde{nm} + \overrightarrow{mb_1}$ would violate our lemma 2.9. Pick from the set $J_2 \cap \overrightarrow{nb_2}$ an element k with the largest possible second coordinate and an element j with the smallest possible second coordinate.

Take x to be the middlepoint of the segment $\overrightarrow{nk}$. By definition of the points n and k we have that $x \in \mathbb{R}^2 \setminus J$.

Now we have our points fixed, so we can begin the proof. We want to show that firstly the point x lies in a bounded component of the complement of our Jordan curve and secondly that no other bounded component exists. Both these claims we achieve by making a counterassumption and creating a contradiction by constructing suitable curves using the points we have fixed above and again applying our corollary 2.9 repeatedly.

1:st claim: The component of $\mathbb{R}^2 \setminus J$ containing the point x is bounded.

Let us make a counterassumption, that the aforementioned component is not bounded. This means that we have a path γ in $\mathbb{R}^2 \setminus J$ that connects the point x and a point y outside our rectangle R. Denote

$$\alpha = \gamma|_{[0,t_0]}, \text{ where } t_0 = \inf\{t \in [0,1] \mid \gamma(t) \in \partial R\}.$$

Now $\alpha(t_0)$ is the first point where the path α meets the boundary of our rectangle.

As we cannot have that $\operatorname{Im} \alpha(t_0) = 0$, for then the path α would cross either a_1 or a_2, we are left with two possibilities.

If $\operatorname{Im} \alpha(t_0) < 0$, then the paths J_2 and

$$\overrightarrow{b_1m} + \widetilde{mn} + \overrightarrow{nx} + \alpha + \widetilde{\alpha(t_0)b_2},$$

where $\widetilde{\alpha(t_0)b_2}$ is the shortest path along the set ∂R between the points $\alpha(t_0)$ and b_2, create a contradiction with lemma 2.9.

If, on the other hand, $\operatorname{Im} \alpha(t_0) > 0$, then the paths J_1 and

$$\overrightarrow{b_2x} + \alpha + \widetilde{\alpha(t_0)b_1},$$

where $\widetilde{\alpha(t_0)b_1}$ is the shortest path along the set ∂R between the points $\alpha(t_0)$ and b_1, create a contradiction with lemma 2.9.

Thus we must have that the point x lies in a bounded component of $\mathbb{R}^2 \setminus J$.

2:nd claim: The component of $\mathbb{R}^2 \setminus J$ containing the point x is the only bounded component of $\mathbb{R}^2 \setminus J$.

If we had another bounded component, say G in $\mathbb{R}^2 \setminus J$, then we see that the path

$$\beta = \overrightarrow{b_1 m} + \widetilde{mn} + \overrightarrow{nk} + \widetilde{kj} + \overrightarrow{jb_2}$$

does not lie in this component, because

- The segments $\overrightarrow{b_1 m}$ and $\overrightarrow{jb_2}$ are contained in the union of J and the unbounded component of its complement.
- The simple arcs $\widetilde{mn}$ and $\widetilde{kj}$ are contained in J.
- The segment $\overrightarrow{nk}$ is contained in the union of J and the component containing the point x.

Note that the set G cannot contain neither of the points a_1 or a_2, and the image of the path β is a compact set, so we find neighbourhoods U_i of a_i, such that $U_i \cap \beta = \emptyset$, where $i = 1, 2$. But by the lemma 2.11, we must have that $a_i \in \partial G$. Thus we especially find points $x_i \in U_i$ such that $x_i \in G, i = 1, 2$. As the component G is necessarily path connected, we find a path α in G which connects the points x_1 and x_2. Now the paths β and $\overrightarrow{a_1 x_1} + \alpha + \overrightarrow{x_2 a_2}$ bring a contradiction with the lemma 2.9.

Thus there exists exactly on bounded component of $\mathbb{R}^2 \setminus J$, so the JCT holds. $\square$

3 Final notions

As promised, we will now look at some other methods of proving the JCT and give the appropriate references. I would also like to note that [DT] contains history, references and a proof of the JCT. In all fairness, I have *not* read all the given references thoroughly, so my summaries can (and propably do) have shortcomings. I apologize to any authors whom I might have miscited or -summarized.

The proof relaying on Brouwer fixed point theorem used only methods of 'planar topology' after the fixed point theorem was proven. Thus in order to construct a proof of the JCT with set prequisites, one can (for example) use a proof of the fixed point theorem that uses some other methods. In this essay we proved the Brouwer fixed point theorem via methods of function theory. The Brouwer fixed point method can be proved in various different ways. I list here a few of them.

- Proof using basic algebraic topology can be found from [Hat].
- Proof using very basic game theory, namely whether a winning strategy exists in the game *hex* can be found from [Ga] (This proof can be followed and understood by highschoolers if the teacher is up to the task.)
- The proof used in this essay basically from [Ma].

As mentioned, algebraic topology would have been in some details of the proofs a very natural tool. The whole theorem can be in fact be seen as a special case of a theorem called the Alexander duality. In his paper [Do] A. Dold guides anyone familiar with basic homology theory throught the proof in the case of subsets of $\mathbb{R}^n$.

There are also proofs that I would classify as 'other proofs'.

- In [Hal] T. Hales defends and represents the original proof of the JCT published by C. Jordan.
- In [Hal2] T. Hales talks about formal proofs in general and especially of the JCT.
- In [Na] Louis Narens gives a nonstandard proof of the JCT. According to [Hal] this is somewhat similar to Jordans original proof.
- In [BJMR] a constructive proof is given. (Constructive in the sense that existence is not enough, the presentation is also essential.)

As mentioned, the stronger Jordan-Schöenflies -theorem can be proved from the JCT. In [So] this is done by improving and applying the Carathodory theorem concerning continuation of conformal maps up to boundary. In [Si] Laurent Siebenmann proves the Jordan-Schöenflies theorem with quite elementary methods, although some algebraic topology is used.

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